energy level ($F = \sigma T^{*4}$) is small; then provided $\tau = \tau^*$ we will have

$$k^* \frac{\partial i}{\partial \tau}\bigg|_{\tau=\tau^*-0} \approx k_T \frac{\partial i}{\partial \tau}\bigg|_{\tau=\tau^*+0} \tag{12}$$

where k_T is the ordinary heat conduction coefficient. Obviously, these stipulations may be posed also with small values of τ^*, when surface cooling occurs.

The viscous layer generates heat which is transferred to the wall and the thermal layer according to the equation

$$q = -\frac{1}{2}\frac{\partial}{\partial x}\int_0^\infty \rho_0 u(u^2 - u_\infty{}^2)dy \tag{13}$$

and which may be found by calculations for an incompressible viscous layer.[2] The thermal stream toward the wall may then be determined from the formula

$$q_0 = q - q_T = q - \frac{k^*}{c_p}\frac{\partial i}{\partial y}\bigg|_{y=0} \tag{14}$$

From these conditions, we may find the parameters of stream flow for different allowable values of q_0 and i_0. In particular, when $q_0 = 0$ we have a thermally insulated plate.

Figure 3 gives curves, which were calculated with the help of high-speed digital computer for case a for various values of β and n.

4. We conclude with a discussion of a process of the same nature which can occur during longitudinal stabilization of a gas discharge in the presence of a large pressure when the magnetic Reynold's number is large.[3]

In this case, the magnetic Reynold's number has a role which is precisely analogous to that of an ordinary Reynold's number in the case analyzed in the foregoing: compressed by a large convective stream which is a good internal conductor of electric current, heat sources will be concentrated near the discharge axes in a narrow region, having a thickness of the order of $\delta \approx L/\sqrt{Re\ m}$; further away from the axes will extend a wider region of the thermal layer, containing no currents and described by Eq. (7). Heat generated by the electric current layer $q = 4\pi\int_0^\infty \nu_m j^2 dy$ (in the flat case) will be transferred to the thermal layer. In this expression j is the electrical current density and V the magnetic viscosity.

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Integral Kinetic Equations of the Theory of Monatomic Gases in the Presence of an External Field of the Forces of Mass

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INTEGRAL kinetic equations of the theory of monatomic rarefied gases were derived in Refs. 1 and 2. Problems in the aerodynamics of monatomic rarefied gases were also stated in Ref. 2.

The integral kinetic equations given in Refs. 1 and 2 were of the following form:

$$f(\bar{r},\bar{u},t) = (1/|u_n|)\,\tilde{\Phi}(\bar{r}_s,\bar{u},t)\Pi(\bar{r},\bar{u},t,\tau_s) + \int_{\tau_s}^{t}\Phi(\bar{r} - \bar{u}(t-\tau),\bar{u},\tau)\Pi(\bar{r},\bar{u},t,\tau)d\tau \tag{1}$$

$$\Pi(\bar{r},\bar{u},t,\tau) = e^{-\int_\tau^t\left[\iiint_{-\infty}^{+\infty}|\bar{u}-\bar{u}_1|\sigma(|\bar{u}-\bar{u}_1|)f(\bar{r}-\bar{u}(t-q),\bar{u}_1,q)d\omega_1\right]dq} \tag{2}$$

$$\Phi(\bar{r},\bar{u},t) = \frac{1}{2}\iiiint\!\!\iint_{-\infty}^{+\infty}|\bar{u}_1 - \bar{u}_2|\,\sigma(|\bar{u}_1 - \bar{u}_2|)f(\bar{r},\bar{u}_1,t) \times f(\bar{r},\bar{u}_2,t)T(\bar{u}_1,\bar{u}_2,\bar{u})d\omega_1 d\omega_2 \tag{3}$$

$$\tilde{\Phi}(\bar{r}_s,\bar{u},t) = \iiint_{(u_1)_n<0} f(\bar{r}_s,\bar{u}_1,t)\,|(u_1)_n|\tilde{T}\,(\bar{u}_1,\bar{n},\bar{u},\theta)d\omega_1 \tag{4}$$

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In these equations f is the distribution function; Π the probability of free motion; Φ the internal generation function; $\tilde{\Phi}$ the boundary generation function; T the internal shock transformant; $\tilde{T}$ the boundary shock transformant; σ the collision cross section, dependent on the relative velocity of colliding particles; $\bar{u}_1$ and $\bar{u}_2$ the integration variables (vector) in velocity space; $d\omega_1$ and $d\omega_2$ are the volume elements in velocity spaces $\bar{u}_1$ and $\bar{u}_2$; τ and q are scalar parameters; $\bar{n}$ is the external normal to the surface of the streamlined body in the point with radius vector $\bar{r}_s$ at the moment of time τ_s; θ the surface temperature in the point with radius vector $\bar{r}_s$ at the moment of time τ_s; and $\bar{r}_s$ and τ_s are certain functions of $\bar{r}$ and t. The remaining notations and definitions are assumed known from Refs. 1 and 2.

It was assumed in Refs. 1 and 2 that the external field of the forces of mass does not affect the moving gas. The present paper determines a system of equations which is a generalization of Eqs. (1–4) and is applicable to the case of motion of the gas when it is affected by a constant external field of the force of mass.

However, due to the exponential decrease of the probability of free motion Π, the equations derived in the present paper are valid also in variable fields of force if the variation of the fields is negligibly small at distances of the order of magnitude of 5–10 mean free paths during periods of time equal to 5–10 average time intervals between the collision of atoms.

The derivation of Eqs. (1–4) implies that the validity of (3) and (4) is not dependent on the assumed absence of the forces of mass. It is the object of this paper, in principle, to modify the derivation of Eqs. (1) and (2), taking into account that the external field of the forces of mass of the atomic trajectories are curvilinear.

It is possible to approach the derivation of these equations in two ways.

First, it is possible to include in the investigation a coordinate system moving at uniform speed, to have in this system of coordinates acceleration $\bar{g}$ equal to zero, to write (1) and (2) in this coordinate system for bodies in motion, and then to return to the previously used coordinates.

Second, it is possible to generalize directly the scheme discussed in Ref. 2.

Since the equations derived herein are applicable to the more interesting case of the motion of gas around a system of fixed bodies, the second approach to the derivation of the equations sought appears more suitable.

1. Probability of Free Motion

Let the atom be in the point with radius vector $\bar{r}$ and have velocity $\bar{u}$ at the moment of time t. Let $\bar{g}$ denote the vector

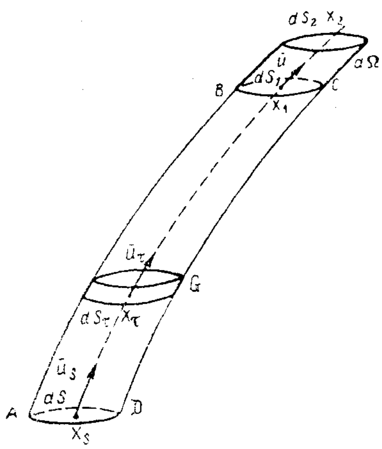

Fig. 1.

of acceleration in the constant field of forces of mass. In the case of free motion of the atom during the interval (τ,t), its position and velocity at moment $q(\tau \leq q \leq t)$ are determined by radius vector $\bar{r}_q$ and vector $\bar{u}_q$ and

$$\bar{r}_q = \bar{r} - \bar{u}(t - q) + \bar{g}[(t - q)^2/2]$$
$$\bar{u}_q = \bar{u} - \bar{g}(t - q) \qquad (5)$$

We now divide interval (τ,t) in elementary intervals Δq_i. Let us denote

$$\tau + \sum_{k=1}^{i} \Delta q_k = q_i \qquad (6)$$

In agreement with (1) the probability that the atom in the point with radius vector $\bar{r}_i$ and velocity $\bar{u}_i$ will collide with another atom during Δq_i is

$$Q_i = \Delta q_i \iiint |\bar{u}_i - \bar{u}'| \, \sigma(|\bar{u}_i - \bar{u}'|)f(\bar{r}_i, \bar{u}', q_i)d\omega' \qquad (7)$$

Then probability Π of free motion during the interval (τ,t) of an atom which at moment τ is in the point with radius vector $\bar{r}_\tau$, has the velocity $\bar{u}_\tau$, and at moment t appears to be in the point with radius vector $\bar{r}$ and to have the velocity $\bar{u}$ is

2. Distribution Function Expressed through Φ, $\tilde{\Phi}$, and Π

Methods similar to the ones investigated in Ref. 1 are studied. Quantity dn, which gives the number of atoms at moment t in volume $d\Omega$, which have velocities from $d\omega$, can be written as follows:

$$dn = f(\bar{r},\bar{u},t)d\Omega d\omega \qquad (9)$$

On the other hand, it is possible to derive the expression for this quantity dn employing functions Φ and $\tilde{\Phi}$ introduced in Ref. 1. The relationships sought are obtained by equating these two expressions.

Let X_1 be the point for the vicinity of which quantity dn is computed (see Fig. 1). Radius vector $\bar{r}_1$ of point X_1 is denoted with $\bar{r}$. We draw the trajectory of the atom through point X_1; when the atom moves along this trajectory it will be at point X_1 at moment t and velocity $\bar{u}$. This trajectory intersects the fixed surface of the streamlined body at point X_s with radius vector $\bar{r}_s$. Equalities (5) give the values of vector $\bar{r}_s$ and velocity $\bar{u}_s$ for $q = \tau_s$. If the constructed trajectories do not meet the surfaces of the streamlined body, it is assumed that X_s lies on it at an infinite distance. We construct the element of the fixed surface dS near point X_s. It can readily be seen that the particles emerging at moment τ_s from the different points of element dS at identical velocities will fill the areas dS_τ, in the following moments τ, which are parallel and equal to the initial element of surface dS.

Volume element $d\Omega$ must be constructed in the vicinity of X_1 in order to compute quantities dn. We select an interval dT. We select an oblique cylinder as the volume element $d\Omega$, for which the lower base dS_1 is constructed near point X_1 and the top base dS_2 is parallel and equal to element dS. The element of the cylinder $d\Omega$ is selected parallel and equal to the segment of the trajectory of the atom which, passing through the lower base dS_1 at the moment of time $(t - dT)$ at velocity $\bar{u}_{t-dT}$, reaches the top base of the cylinder dS_2 with velocity $\bar{u}$ at moment t.

Conservation of small quantities not higher than the first order of smallness in relation to dT is advantageous in all calculations. Therefore, further calculations take into account the first powers of dT, neglecting the higher powers of small quantities. In such a case the height of the cylinder $dH = |(u_{t-dT})_n|dT$ can be assumed to equal $dH = |u_n|dT$ and the volume $d\Omega = |u_n|dS \cdot dT$.

The cylinder elements $d\Omega$, in this case, are directed along $\bar{u}$, and the trajectories of atoms which at moment t reach in the corresponding points dS_1 and dS_2 with velocity $\bar{u}$ will coincide.

We first compute the number dn_1 of particles reaching $d\Omega$ from the curvilinear cylinder $ABCD$ with bases dS and dS_1, the elements of which are the atom trajectories emerging from the boundaries dS at velocity $\bar{u}_s$. We select point X_τ on trajectory X_sX_1 and construct near it a volume—disk G with base dS_τ and height dh. The particles emerging from dS_τ at moment τ and velocity $\bar{u}_\tau$ in the case of this free movement at moment t reach section dS at velocity $\bar{u}$. The particles that belong to volume G, which at moment t reach the upper base of volume $d\Omega$–dS_2 with velocity $\bar{u}$, must emerge from dS_τ at the moment of time $(\tau - d\tilde{T})$ with velocity $\bar{u}_{\tau-d\tilde{T}}$. The magnitude of interval $d\tilde{T}$ is related to dT through

$$|(u_\tau)_n|d\tilde{T} = |u_n|dT = dH \qquad (10)$$

Computing the number of atoms which disk G delivers to volume $d\Omega$ during the interval $d\tilde{T}$, one can neglect the variation of the generation function Φ and the probability of free motion Π during this interval because consideration of these quantities gives magnitude of a higher order of smallness.

$$\Pi(\bar{r},\bar{u},t,\tau) = e^{-\int_\tau^t \left[\iiint_{-\infty}^{+\infty} |\bar{u} - \bar{g}(t-q) - \bar{u}'| \sigma(|\bar{u} - \bar{g}(t-q) - \bar{u}'|) \cdot (\bar{r} - \bar{u}(t-q) + \bar{g}[(t-q)^2/2], \bar{u}'q) d\omega' \right] dq} \qquad (8)$$

Thus, for the complete interval $d\tilde{T}$ in disk G, functions Φ and Π are considered equal to $\Phi(\bar{r}_\tau,\bar{u}_\tau,\tau)$ and $\Pi(\bar{r},\bar{u},t,\tau)$, respectively. Then the number of atoms dn_τ supplied by disk G during the time $d\tilde{T}$ to volume $d\Omega$ with the necessary velocities equals

$$dn_\tau = dS_\tau \cdot dh \cdot d\omega_\tau \cdot \Phi(\bar{r}_\tau,\bar{u}_\tau,\tau)\cdot \Pi(\bar{r},\bar{u},t,\tau)d\tilde{T} \qquad (11)$$

$$\Pi(\bar{r},\bar{u},t,\tau) = e^{-\int_\tau^t \left[\iiint_{-\infty}^{+\infty} |\bar{u}-\bar{g}(t-q)-\bar{u}'|\cdot\sigma(|\bar{u}-\bar{g}(t-q)-\bar{u}'|)\cdot f(\bar{r}-\bar{u}(t-q)+\bar{g}[(t-q)^2]/2,\bar{u}',q)d\omega'\right]dq} \qquad (20)$$

Replacing $d\tilde{T}$ with its expression through dT from (10), considering the equalities

$$dS_\tau = dS \qquad d\Omega = |u_n|dSdT \qquad d\omega_\tau = d\omega \qquad (12)$$

and selecting τ as the independent variable, introducing the notation $dh/|(u_\tau)_n| = d\tau$, we have

$$dn_\tau = d\Omega d\omega \Phi(\bar{r}_\tau,\bar{u}_\tau,\tau)\Pi(\bar{r},\bar{u},t,\tau)d\tau \qquad (13)$$

The over-all number of particles, of interest in this case, which is supplied by cylinder $ABCD$ to volume $d\Omega$ is derived by integrating (13) according to τ from τ_s to t:

$$dn = d\Omega d\omega \int_{\tau_s}^t \Phi(\bar{r}_\tau,\bar{u}_\tau,\tau)\Pi(\bar{r},\bar{u},t,\tau)d\tau =$$
$$d\Omega d\omega \int_{\tau_s}^t \Phi[\bar{r} - \bar{u}(t-\tau) + \bar{g}\,[(t-\tau)^2/2,\ \bar{u} -$$
$$\bar{g}(t-\tau),\ \tau]\cdot\Pi(\bar{r},\bar{u},t,\tau)d\tau \qquad (14)$$

We now compute the number of particles dn_2 supplied by surface element dS to volume $d\Omega$ at moment of time t. At moment of time t the volume $d\Omega$ contains those particles with velocities from $d\omega$ which have been generated on surface element dS during the interval $(\tau_s - dT_s,\tau_s)$ and which have passed the total path without collisions. With the assumed accuracy, the number of such particles is given by

$$dn_2 = dS\cdot d\omega_s\cdot\tilde{\Phi}(\bar{r}_s,\bar{u}_s,\tau_s)\cdot\Pi(\bar{r},\bar{u},t,\tau_s)dT_s \qquad (15)$$

Introducing quantity dT instead of dT_s, in agreement with (10) and in consideration of (12), the equation for dn_2 is written as follows:

$$dn_2 = (1/|(u_s)_n|)\tilde{\Phi}(\bar{r}_s,\bar{u}_s,\tau_s)\cdot\Pi(\bar{r},\bar{u},t,\tau_s)d\Omega d\omega \qquad (16)$$

Substituting (9, 14, and 16) in

$$dn = dn_1 + dn_2 \qquad (17)$$

we find the relationship sought between functions $f,\Phi,\tilde{\Phi}$, and Π:

$$f(\bar{r},\bar{u},t) = (1/|(u_s)_n|)\tilde{\Phi}(\bar{r}_s,\bar{u}_s,\tau_s)\cdot\Pi(\bar{r},\bar{u},t,\tau_s) +$$
$$\int_{\tau_s}^t \Phi[\bar{r} - \bar{u}(t-\tau) + \bar{g}[(t-\tau)^2/2],\ \bar{u} -$$
$$\bar{g}(t-\tau),\tau]\cdot\Pi(\bar{r},\bar{u},t,\tau)\ d\tau \qquad (18)$$

3. General Remarks

The following general remarks are made in conclusion.

First of all, the foregoing implies clearly that the system of kinetic equations generalizing (1–4) in the case of motion of a gas in a constant external field of force is as follows:

$$f(\bar{r},\bar{u},t) = (1/|(u_s)_n|)\tilde{\Phi}(\bar{r}_s,\bar{u}_s,\tau_s)\cdot\Pi(\bar{r},\bar{u},t,\tau_s) +$$
$$\int_{\tau_s}^t \Phi[\bar{r} - \bar{u}(t-\tau) + \bar{g}[(t-\tau)^2/2],\ \bar{u} -$$
$$\bar{g}(t-\tau),\tau]\cdot\Pi(\bar{r},\bar{u},t,\tau)d\tau \qquad (19)$$

$$\Phi(\bar{r},\bar{u},t) = \frac{1}{2}\iiiint\!\!\iint_{-\infty}^{+\infty} |\bar{u}_1 - \bar{u}_2|\ \sigma(|\bar{u}_1 - \bar{u}_2|)\times$$
$$f(\bar{r},\bar{u}_1,t)f(\bar{r},\bar{u}_2,t)\times T(\bar{u}_1,\bar{u}_2,\bar{u})d\omega_1 d\omega_2 \qquad (21)$$

$$\tilde{\Phi}(\bar{r}_s,\bar{u},t) = \iiint_{(u_1)_n<0} f(\bar{r}_s,\bar{u}_1,t)|(u_1)_n|\tilde{T}(\bar{u}_1,\bar{n},\bar{u},\theta)d\omega_1 \qquad (22)$$

As in the system of Eqs. (1–4), functions Π, Φ, and $\tilde{\Phi}$ can be omitted from the system of Eqs. (19–22) and an integral equation can be derived for the function f as follows:

$$f = V(f) \qquad (23)$$

where $V(f)$ is a certain integral operator over f.

Furthermore, it is obvious that the statement of the problems of aerodynamics for the system of Eqs. (19–22) or for Eq. (23) is practically the same as the statement of the problems for the system (1–4) or its corresponding integral equation.

If one considers the external problems formulated in Ref. 2, then the total variation is reduced to the circumstance that at infinity one has the Boltzmann distribution instead of the Maxwell distribution, which corresponds to the existing field of the forces of mass. The validity of the Boltzmann distribution can be readily established directly from (23) written for the steady state of a gas which fills the complete space.

As in Ref. 1, it is possible to determine that the Boltzmann equation follows from (19–22), which in the case under investigation and in our notations has the following form:

$$\frac{\partial f}{\partial t} + u_1\frac{\partial f}{\partial x_1} + u_2\frac{\partial f}{\partial x_2} + u_3\frac{\partial f}{\partial x_3} + g_1\frac{\partial f}{\partial u_1} + g_2\frac{\partial f}{\partial u_2} + g_3\frac{\partial f}{\partial u_3} =$$
$$\Phi(\bar{r},\bar{u},t) - f(\bar{r},\bar{u},t)\iiint_{-\infty}^{+\infty} |\bar{u}-\bar{u}'|\ \sigma(|\bar{u}-\bar{u}'|)f(\bar{r},\bar{u}',t)d\omega' \qquad (24)$$

Finally, it is clear that the system of Eqs. (19–22) can be solved by the method of sequential approximations, specified in Ref. 2 for the solution of the system of Eqs. (1–4).

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